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Where do homogeneous polynomials on ℓ_1^n attain their norm?☆

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Abstract

Using a 'reasonable' measure in $\mathscr{P}(^2\ell_1^n)$, the space of 2-homogeneous polynomials on ℓ_1^n, we show the existence of a set of positive (and independent of n) measure of polynomials which do not attain their norm at the vertices of the unit ball of ℓ_1^n. Next we prove that, when n grows, almost every polynomial attains its norm in a face of 'low' dimension.

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1. Introduction, notation and definitions

In the past few years there has been an increasing interest, within the theory of polynomials in Banach spaces, in the study of the geometry of the spaces of polynomials (see, for instance, [1,3–7]).

In this direction, in the conference 'Function Theory on Infinite Dimensional Spaces VII', held in Madrid in 2001, Professor Zalduendo asked the question of 'how many' homogeneous polynomials will attain their norm at the vertices of the unit

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ball of ℓ_∞^n when n tends to infinity. He conjectured that 'almost everyone'. In this direction, he and Carando published recently a paper giving qualitative general results (see [2]). As they say in the introduction, the question is to study how likely it is for a polynomial $P: E \to \mathbb{R}$ to attain its norm at a given subset A of the unit ball B_E. In our paper, we give quantitative results referring to 2-homogeneous polynomials on ℓ_1^n, as an example of the results that can be expected in more general cases. We use normalized Lebesgue's measure μ_n on the unit ball of the space $\mathscr{L}_s(^2\ell_1^n)$ of symmetric bilinear forms to count 'how many' polynomials attain their norm wherever. The reason for using this measure, instead of normalized Lebesgue's measure on the polynomial unit ball is that it is (by far) easier to deal with. On the other hand, it is also a reasonable measure since, by the polarization formula, for every 2-homogeneous polynomial P on ℓ_1^n, we have that $||P|| \leqslant ||A|| \leqslant 2||P||$, where A is the associated symmetric bilinear form.

The first result we have is that Zalduendo's conjecture fails in this setting (see Theorem 3). This is not so surprising since the number of vertices in the unit ball of ℓ_1^n is just $2n$, whereas in the unit ball of ℓ_∞^n there are 2^n vertices. The main result (Theorem 4), however, shows that even in this case Zalduendo's conjecture is not far from the truth, in the sense that, asymptotically, almost every polynomial attains its norm in a face of 'low' dimension.

The notation will be the usual in this context. E will denote a finite-dimensional Banach space. Associated to it, we are going to consider its unit ball B_E, the space of real-valued 2-homogeneous polynomials $\mathscr{P}(^2E)$, and the space of real-valued symmetric bilinear forms $\mathscr{L}_s(^2E)$. Given a polynomial P, we are going to write A for the unique symmetric associated bilinear form. We are going to consider only polynomials P such that $A \in B_{\mathscr{L}_s(^2E)}$. ℓ_1^n will be the Banach space $(\mathbb{R}^n, ||\cdot||_1)$, ℓ_∞^n will be $(\mathbb{R}^n, ||\cdot||_\infty)$ and $\{e_i\}_{i=1}^n$ will denote the canonical basis of $\mathbb{R}^n$.

We identify $\mathscr{L}_s(^2\ell_1^n)$ with $\ell_\infty^{\frac{n(n+1)}{2}}$ via the isometry $A \mapsto (A(e_i, e_j) = a_{ij})_{1 \leqslant i \leqslant j}^n$. With this, denoting the natural identification of 2-homogeneous polynomials and the corresponding symmetric bilinear forms as $F: P \mapsto A$, for any measurable subset $\mathscr{S} \subset \mathscr{P}(^2\ell_1^n)$ one defines

$$\mu_n(\mathscr{S}) := 2^{-\frac{n(n+1)}{2}} \lambda_{\frac{n(n+1)}{2}}(F(\mathscr{S})),$$

where λ_d is the usual Lebesgue measure in $\mathbb{R}^d$.

For a general definition of a vertex and an m-dimensional face of a convex polytope, we refer the reader to [8]. Here all we are going to use is that [8, pp. 55–56] in $B_{\ell_1^n}$, the vertices are just $\pm e_i$, $i = 1, \dots, n$, and an $(m-1)$-dimensional face (or $(m-1)$-face) is just the convex hull of m linearly independent vertices. The interior of an m-face C is the set of points of C that are not in any k-face, for $k < m$.

Though we are not going to say it from now on, it is not difficult to show that all the sets we are going to consider are measurable.

2. The results

Lemma 1. *Let E be a normed vector space, let $P \in \mathscr{P}(^2E)$ and let $T \in \mathscr{L}_s(^2E)$ be its associated symmetric bilinear form. Suppose $x, y \in E$ and suppose $|P(x)| \geqslant |P(y)|$.*

(i) *If $|P(x)| \geqslant |T(x,y)|$ then, for every $0 < \lambda < 1$, we have that $|P(\lambda x + (1-\lambda)y)| \leqslant |P(x)|$.*
(ii) *Conversely, if $|T(x,y)| > |P(x)|$ and $P(x)$ and $T(x,y)$ have the same sign, then there exists $\lambda \in (0,1)$ such that $|P(\lambda x + (1-\lambda)y)| > |P(x)|$.*

Proof. Let us suppose first that $|P(x)| \geqslant |T(x,y)|$. Then, for every $\lambda \in (0,1)$,

$$\begin{aligned} |P(\lambda x + (1-\lambda)y)| &= |\lambda^2 P(x) + (1-\lambda)^2 P(y) + 2\lambda(1-\lambda)T(x,y)| \\ &\leqslant |\lambda^2 P(x)| + |(1-\lambda)^2 P(y)| + |2\lambda(1-\lambda)T(x,y)| \\ &\leqslant |P(x)|, \end{aligned}$$

because $\lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) = 1$.

Conversely, suppose that $T(x,y) > P(x) \geqslant 0$ (the other case is similar). Let

$$f(\lambda) = P(\lambda x + (1-\lambda)y) = \lambda^2 P(x) + (1-\lambda)^2 P(y) + 2\lambda(1-\lambda)T(x,y).$$

Then

$$f'(\lambda) = 2(\lambda P(x) + (\lambda - 1)P(y) + (1-2\lambda)T(x,y))$$

and $f'(\lambda) = 0$ only when

$$\lambda = \lambda_0 = \frac{P(y) - T(x,y)}{P(x) + P(y) - 2T(x,y)}.$$

Clearly $0 < \lambda_0 < 1$ and, since $f''(\lambda) = 2(P(x) + P(y) - 2T(x,y)) < 0$, we get that $f(\lambda_0) = P(\lambda_0 x + (1-\lambda_0)y) > P(x)$. Moreover, we get that

$$f(\lambda_0) = \frac{P(y)P(x) - T(x,y)^2}{P(x) + P(y) - 2T(x,y)}. \qquad \square$$

As an application of the first part of the lemma, we have the following:

Proposition 2. *Let $P \in \mathscr{P}(^2\ell_1^n)$, let $A \in \mathscr{L}_s(^2\ell_1^n)$ be its associated symmetric bilinear form and let $i \in \{1, \dots, n\}$ be such that $|P(e_i)| \geqslant |P(e_j)|$ for every $j \in \{1, \dots, n\}$. Suppose that, for every $j \in \{1, \dots, n\}$, $|P(e_i)| \geqslant |A(e_i, e_j)|$. Then P attains its norm either at e_i or at one of the $(n-2)$-dimensional faces not adjacent to e_i or $-e_i$.*

Proof. Let us suppose without loss of generality that $i = 1$. A point y in one of the non-adjacent $(n-2)$-dimensional faces can always be written in the form

$y = \sum_{j=2}^n \alpha_j e_j$, where $\sum_{j=2}^n |\alpha_j| = 1$. Let us note that

$$|A(e_1, y)| = \left| \sum_{j=2}^n \alpha_j A(e_1, e_j) \right| \leqslant \sum_{j=2}^n |\alpha_j| |A(e_1, e_j)| \leqslant |P(e_1)|.$$

So, consider any point z in the unit ball of ℓ_1^n. There exists y in one of the $(n-2)$-dimensional faces not adjacent to e_1 or $-e_1$ and $\lambda \in [0,1]$ such that $z = \lambda e_1 + (1-\lambda) y$. If $|P(e_1)| \geqslant |P(y)|$, we can use Lemma 1(i) to prove that $|P(z)| \leqslant |P(e_1)|$. If $|P(e_1)| \leqslant |P(y)|$ we use again Lemma 1(i) to prove that $|P(z)| \leqslant |P(y)|$. □

We can also use the second part of Lemma 1 to prove the next theorem.

Theorem 3 (Failure of Zalduendo's conjecture for ℓ_1^n). *For any* $n \geqslant 2$, *if we denote* $C = \{P \in \mathscr{P}(^2\ell_1^n)$ *such that* $\|A\| \leqslant 1$ *and P does not attain its norm at a vertex*$\}$, *then* $\mu_n(C) \geqslant \frac{1}{6}$.

Proof. We define the following sets:

$$B := \left\{ P \text{ such that there exist } i_0, j_0 \text{ with} \left\{ \begin{array}{c} \max_i |P(e_i)| = |P(e_{i_0})| \\ |P(e_{i_0})| < |A(e_{i_0}, e_{j_0})| \end{array} \right\} \right\},$$

$$\hat{B} := \left\{ P \text{ such that there exist } i_0, j_0 \text{ with} \left\{ \begin{array}{c} \max_i |P(e_i)| = |P(e_{i_0})| \\ |P(e_{i_0})| < |A(e_{i_0}, e_{j_0})| \\ \operatorname{sign} P(e_{i_0}) \neq \operatorname{sign} A(e_{i_0}, e_{j_0}) \end{array} \right\} \right\},$$

$$\tilde{B} := \left\{ P \text{ such that there exist } i_0, j_0 \text{ with} \left\{ \begin{array}{c} \max_i |P(e_i)| = |P(e_{i_0})| \\ |P(e_{i_0})| < |A(e_{i_0}, e_{j_0})| \\ \operatorname{sign} P(e_{i_0}) = \operatorname{sign} A(e_{i_0}, e_{j_0}) \end{array} \right\} \right\}.$$

Let us consider the linear isometry $\ell_\infty^{\frac{n(n+1)}{2}} \to \ell_\infty^{\frac{n(n+1)}{2}}$ given by $(a_{ij})_{j \geqslant i} \mapsto (\tilde{a}_{ij})_{j \geqslant i}$, where $\tilde{a}_{ij} = -a_{ij}$ if $j > i$ and $\tilde{a}_{ii} = a_{ii}$. Clearly the image of $\tilde{B}$ is just $\hat{B}$. Using the change of variables theorem, we obtain that $\mu_n(\tilde{B}) = \mu_n(\hat{B})$. Besides, $B = \tilde{B} \cup \hat{B}$ and, by Lemma 1(ii), $\tilde{B} \subset C$. Therefore

$$\mu_n(C) \geqslant \frac{\mu_n(B)}{2}.$$

Now, using the usual identification $P \leftrightarrow (A(e_i, e_j) = a_{ij})_{j \geqslant i}$, we have that

$$B^c \subset \bigcup_{k=1}^n \left\{ |a_{kk}| = \max_{1 \leqslant i \leqslant n} |a_{ii}| \text{ and } |a_{kk}| = \max_{1 \leqslant j \leqslant n} |a_{kj}| \right\},$$

where we take $a_{kj} = a_{jk}$ if $k > j$.

For each $k = 1, \ldots, n$, the measure of the set $\{|a_{kk}| = \max_i |a_{ii}|$ and $|a_{kk}| = \max_j |a_{kj}|\}$ can be calculated easily by integration to be $\frac{1}{2n-1}$. Therefore we have that

$$\mu_n(C) \geqslant \frac{\mu_n(B)}{2} = \frac{1-\mu_n(B^c)}{2} \geqslant \frac{1-\frac{n}{2n-1}}{2} = \frac{n-1}{4n-2} \geqslant \frac{1}{6},$$

for every $n \geqslant 2$. $\square$

This result shows the existence of a set of positive measure of polynomials which do not attain their norm at the vertices. We are reasonably sure of the existence of another set of positive (and independent of n) measure of polynomials which do attain their norm at the vertices, but we have not been able to prove this yet.

Indeed, it seems to be the case that 'most' of the polynomials $P \in \mathscr{P}(^2\ell_1^n)$ attain their norm in the low-dimensional faces. This is the content of our next (and main) theorem.

Theorem 4. *Let S_n^m be the set of polynomials $P \in \mathscr{P}(^2\ell_1^n)$ such that $||A|| \leqslant 1$ and P attains its norm in the interior of an $(m-1)$-face. Then*

$$\lim_{n\to\infty} \mu_n\left(\bigcup_{m>16\sqrt{n}} S_n^m\right) = 0. \tag{1}$$

The idea behind the proof of Theorem 4 is to find sets B_n^m such that $S_n^m \subset B_n^m$, each B_n^m is 'easy' to measure, and condition (1) still holds for B_n^m. To do this we need some preliminary results.

Proposition 5. *If P is a polynomial that attains its maximum in the interior of the $(m-1)$-face C given by the vertices $v_1, \dots, v_m$, and if $P(v_1) \leqslant P(v_2) \leqslant \cdots \leqslant P(v_m)$, then*

$$\begin{aligned} P(v_1) &\leqslant A(v_1, v_j) \quad \forall j > 1, \\ P(v_2) &\leqslant A(v_2, v_j) \quad \forall j > 2, \\ &\vdots \\ P(v_{m-1}) &\leqslant A(v_{m-1}, v_m). \end{aligned}$$

Proof. The interior of C is given by

$$int(C) = \{\lambda_1 v_1 + \cdots + \lambda_{m-1} v_{m-1} + (1 - \lambda_1 - \cdots - \lambda_{m-1}) v_m,$$

where

$$\left.\lambda_i \in (0,1) \quad (1 \leqslant i \leqslant m-1) \text{ and } \sum_{i=1}^{m-1} \lambda_i < 1 \right\}.$$

We call $D = \{(\lambda_1, \dots, \lambda_{m-1}) \in (0,1)^{m-1} : \sum_{i=1}^{m-1} \lambda_i < 1\}$ and we define $f : D \to \mathbb{R}$ by

$$f(\lambda_1, \dots, \lambda_{m-1}) = P(\lambda_1 v_1 + \cdots + \lambda_{m-1} v_{m-1} + (1 - \lambda_1 - \cdots - \lambda_{m-1}) v_m).$$

We have that f is the polynomial of degree 2 given by

$$\begin{aligned} f(\lambda_1, \ldots, \lambda_{m-1}) = & \sum_{i=1}^{m-1} \lambda_i^2 P(v_i) + \left(1 + \sum_{i=1}^{m-1} \lambda_i^2 + 2 \sum_{1=i<j}^{m-1} \lambda_i \lambda_j - 2 \sum_{i=1}^{m-1} \lambda_i \right) P(v_m) \\ & + 2 \sum_{1=i<j}^{m-1} \lambda_i \lambda_j A(v_i, v_j) + 2 \sum_{i=1}^{m-1} \lambda_i A(v_i, v_m) \\ & - 2 \sum_{i=1}^{m-1} \lambda_i^2 A(v_i, v_m) - 2 \sum_{i \neq j}^{m-1} \lambda_i \lambda_j A(v_i, v_m). \end{aligned}$$

As f attains its maximum in D, we have that the Hessian matrix $H = (H_{ij})_{i,j=1}^{m-1}$ of f, which is constant, is negative semidefinite. Then, considering $u_{ij} = e_i - e_j$ for $i<j$, we have that

$$\tfrac{1}{2}(H_{ii} + H_{jj} - 2H_{ij}) = \tfrac{1}{2} u_{ij}^t H u_{ij} \leqslant 0.$$

Now,

$$\begin{aligned} \tfrac{1}{2} H_{ii} &= P(v_i) + P(v_m) - 2A(v_i, v_m), \\ \tfrac{1}{2} H_{jj} &= P(v_j) + P(v_m) - 2A(v_j, v_m), \\ \tfrac{1}{2} H_{ij} &= P(v_m) + A(v_i, v_j) - A(v_i v_m) - A(v_j, v_m) \end{aligned}$$

and so

$$P(v_i) + P(v_j) \leqslant 2A(v_i, v_j) \tag{2}$$

holds for $1 \leqslant i < j \leqslant m-1$.

As, in addition, $P(v_i) + P(v_m) - 2A(v_i, v_m) = \frac{1}{2} H_{ii} \leqslant 0$ for $1 \leqslant i \leqslant m-1$, we have that (2) holds for $1 \leqslant i < j \leqslant m$. Using the condition $P(v_1) \leqslant \cdots \leqslant P(v_m)$ it is straightforward to conclude the result. □

The following two lemmas can be easily proved by induction.

Lemma 6. *If $n \geqslant 1$, we have that*

$$\int_{x_{n-1}=x_n}^{1} \cdots \int_{x_0=x_1}^{1} \prod_{j=1}^{n-1} (1-x_j)^j \, dx_0 \cdots dx_{n-1} = \frac{(1-x_n)^{\frac{n(n+1)}{2}}}{\prod_{k=1}^{n} \frac{k(k+1)}{2}}.$$

Lemma 7.

$$\prod_{k=2}^{m} \frac{k(k+1)}{2} = \frac{(m+1)!^2}{2^m (m+1)}.$$

We can use now these lemmas and Proposition 5 to prove

Proposition 8.

$$\mu_n(S_n^m) \leqslant \frac{\binom{n}{m} 2^{2m+1}}{(m+1)!}.$$

Proof. Given an $(m-1)$-face C, we will call M_C (resp. N_C) the set of polynomials $P \in \mathscr{P}(^2\ell_1^n)$ with $||A|| \leqslant 1$ such that P attains its maximum (resp. minimum) in the interior of C. It is trivial that $\mu_n(M_C) = \mu_n(N_C)$.

Let us call C_0 the $(m-1)$-face given by $e_1, \ldots, e_m$. It is not difficult to see that, given any other $(m-1)$-face, say C, there exists a linear isometry $T : \ell_\infty^{\frac{n(n+1)}{2}} \to \ell_\infty^{\frac{n(n+1)}{2}}$ (with $|\det(T)| = 1$) that maps M_C onto M_{C_0}. Using the change of variables theorem, it follows that

$$\mu_n(N_C) = \mu_n(M_C) = \mu_n(M_{C_0}).$$

We also know [8, p. 56] that there are $\binom{n}{m} 2^m$ different $(m-1)$-faces in $B_{\ell_1^n}$. Therefore, we have that

$$\mu_n(S_n^m) \leqslant \sum_C (\mu_n(M_C) + \mu_n(N_C)) = \binom{n}{m} 2^{m+1} \mu_n(M_{C_0}). \tag{3}$$

Now, if we make the convention $a_{ij} = a_{ji}$ if $i > j$ and define, for each permutation $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$, the set B_σ by

$$B_\sigma = \Big\{ A = (a_{ij}) \text{ such that } ||A|| \leqslant 1, \quad a_{\sigma(1),\sigma(1)} \leqslant \cdots \leqslant a_{\sigma(m),\sigma(m)} \text{ and}$$

$$\left\{ \begin{array}{c} a_{\sigma(1),\sigma(1)} \leqslant a_{\sigma(1),\sigma(2)}, \ldots, a_{\sigma(1),\sigma(m)} \\ a_{\sigma(2),\sigma(2)} \leqslant a_{\sigma(2),\sigma(3)}, \ldots, a_{\sigma(2),\sigma(m)} \\ \cdots \\ a_{\sigma(m-1),\sigma(m-1)} \leqslant a_{\sigma(m-1),\sigma(m)} \end{array} \right\} \Big\},$$

we get, using Proposition 5, that $M_{C_0} \subset \bigcup_\sigma B_\sigma$.

But we have as above that $\mu_n(B_\sigma) = \mu_n(B_{id})$ for every σ. Moreover, we have that $2^{\frac{m(m+1)}{2}} \mu_n(B_{id})$ is just

$$\int_{-1}^{1} \int_{a_{22}=a_{11}}^{1} \cdots \int_{a_{mm}=a_{m-1,m-1}}^{1} \prod_{j=1}^{m-1} (1 - a_{jj})^{m-j} da_{mm} \cdots da_{22}\, da_{11}. \tag{4}$$

Now, by Lemma 6, (4) is equal to

$$\frac{1}{\prod_{k=1}^{m-1} \frac{k(k+1)}{2}} \int_{-1}^{1} (1 - a_{11})^{\frac{m(m-1)}{2} + m - 1} da_{11} = 2^{\frac{m(m+1)}{2}} \frac{1}{\prod_{k=1}^{m} \frac{k(k+1)}{2}}$$

and by Lemma 7,

$$\mu_n(B_{id}) = \frac{2^m (m+1)}{(m+1)!^2}.$$

So

$$\mu_n(M_{C_0}) \leqslant m! \mu_n(B_{id}) = \frac{2^m}{(m+1)!}$$

and an appeal to (3) finishes the proof. $\square$

Finally, we need a technical result.

Proposition 9. *There exists a natural number* n_0 *such that, for every* $n \geqslant n_0$, *we have*

$$\sum_{m=8n}^{n^2} \frac{\binom{n^2}{m} 2^{2m}}{m!} \leqslant \frac{1}{n}.$$

Proof. The proof lies in the following two claims:

Claim 1. *There exists a natural number* n_0 *such that, for every* $n \geqslant n_0$, *we have that*

$$\frac{\binom{n^2}{8n} 2^{16n}}{(8n)!} \leqslant \frac{1}{n^3}.$$

Claim 2. *If* $8n \leqslant m \leqslant n^2 - 1$ *and we call*

$$x_m = \frac{\binom{n^2}{m} 2^{2m}}{m!},$$

we have that $x_m \geqslant x_{m+1}$.

With these two claims, if $n \geqslant n_0$ then

$$\sum_{m=8n}^{n^2} \frac{\binom{n^2}{m} 2^{2m}}{m!} \leqslant \sum_{m=8n}^{n^2} \frac{\binom{n^2}{8n} 2^{16n}}{(8n)!} \leqslant \frac{n^2}{n^3} = \frac{1}{n}$$

and we are done.

In order to prove the first claim we call

$$y_n = \frac{\binom{n^2}{8n} 2^{16n} n^3}{(8n)!}.$$

We will see that $\lim_{n\to\infty} y_n = 0$. We have

$$\begin{aligned}\frac{y_{n+1}}{y_n} &= 2^{16}\left(1+\frac{1}{n}\right)^3 \frac{(n^2+1+2n)\cdots(n^2+1)}{(8n+8)^2\cdots(8n+1)^2(n^2+1-6n-8)\cdots(n^2-8n+1)}\\ &= \frac{(1+1/n)^3}{2^{32}}\frac{(n^2+1+2n)\cdots(n^2+2n-6)}{(\frac{8n+8}{8})^2\cdots(\frac{8n+1}{8})^2}\frac{(n^2+2n-7)\cdots(n^2+1)}{(n^2+1-6n-8)\cdots(n^2-8n+1)}\\ &\leqslant \frac{(1+\frac{1}{n})^3}{2^{32}}\frac{(n^2+1+2n)\cdots(n^2+2n-6)}{(\frac{8n+8}{8})^2\cdots(\frac{8n+1}{8})^2}\left(\frac{n^2+2n-7}{n^2-8n+1}\right)^{2n-8} = B_n.\end{aligned}$$

It is easy to see that $\lim_{n\to\infty} B_n = \frac{e^{20}}{2^{32}} < \frac{1}{2}$. Therefore, by the quotient criterium, $\lim_{n\to\infty} y_n = 0$.

To see the second claim, we are going to prove that $\frac{x_{m+1}}{x_m} \leqslant 1$.

We have that $\frac{x_{m+1}}{x_m} = \frac{4(n^2-m)}{(m+1)^2}$. But $\frac{4(n^2-m)}{(m+1)^2} \leqslant 1$ if and only if $m \geqslant 1+\sqrt{4n^2-1}$. As $8n \geqslant 1+\sqrt{4n^2-1}$, we can conclude the result. □

Finally, we can give the proof of Theorem 4.

Proof. We have that

$$\begin{aligned}\mu_n\left(\bigcup_{m>16\sqrt{n}} S_n^m\right) &\leqslant \sum_{m>16\sqrt{n}} \mu_n(S_n^m) \leqslant \sum_{m>16\sqrt{n}} \frac{\binom{n}{m}2^{2m+1}}{(m+1)!}\\ &\leqslant \sum_{m=16[\sqrt{n}]}^{n} \frac{\binom{n}{m}2^{2m}}{m!} \leqslant \sum_{m=8([\sqrt{n}]+1)}^{([\sqrt{n}]+1)^2} \frac{\binom{([\sqrt{n}]+1)^2}{m}2^{2m}}{m!},\end{aligned}$$

where $[\cdot]$ denotes integer part.

Therefore, by Proposition 9, there exists a natural number n_0 such that

$$\mu_n\left(\bigcup_{m>16\sqrt{n}} S_n^m\right) \leqslant \frac{1}{[\sqrt{n}]+1}$$

for every $n \geqslant n_0$, and we are done. □

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